*Int. J. Solids Structures* Vol. 27, No. 8, pp. 999–1010, 1991
Printed in Great Britain.

0020-7683/91 $3.00+.00


# AN EFFICIENT SOLUTION TO THE FREE VIBRATION OF THICK ANGLE-PLY LAMINAE

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(*Received* 12 *November* 1989; *in revised form* 15 *May* 1990)

**Abstract**—A mixed method of elastodynamics, termed the state space approach, is presented for analyzing the dynamic response of composite plates. The method is based on the principle of reducing the governing three-dimensional equations to a set of two-dimensional equations while keeping the field equations in their exact form. Natural frequencies of vibration for thick orthotropic laminae with varying angles of inclination of the orthotropic axes are presented for two distinctly different materials and for large values of thickness. Comparisons with the exact solution of the three-dimensional equations of elasticity establish the effectiveness of the present approach for the class of problems considered. The numerical results from the present theory with a limited number of terms considered in the mathematical expansion in the thickness coordinate are seen to be in excellent agreement with the exact results even for very thick plates.

## INTRODUCTION

The linear elastodynamic behavior of layered and composite plates has received considerable attention with the introduction of various approximate methods for solution in order to circumvent the complexity of an exact solution. With the earliest group of literature pertaining to the general plate analysis, known as classical plate theory, Reissner and Stavsky (1961) have shown that, with the well-known Kirchhoff assumptions, there exists a bending–stretching coupling which remains in spite of linearizing the governing plate equations. In laminated plates, it was also established (Whitney and Leissa, 1969) that the bending–stretching coupling increased the static deflections and reduced the fundamental frequencies. The results of classical plate theory become questionable for thick plates or when the layer properties differ appreciably. Also, since the transverse shear deformations are neglected, all the elastic constants of orthotropy are not taken into account.

In a different class of problems, several authors, including Yang *et al.* (1966) and Pagano (1978), have extended Mindlin's theory to composites and layered plates by means of appropriate shear correction factors to account for transverse shear deformations. The complexity of theories from such extensions, where displacements are approximated by higher order expansions in terms of thickness coordinate, often offsets the accuracy. With reference to layered plates, continuity of tractions at the interfaces between the layers is not satisfied. Also, the governing differential equations of three-dimensional elasticity are not exactly satisfied. Further, as shown by Pagano (1978), the edge boundary conditions in these methods are generally less in number than required by the order of the associated differential equations. It can be safely said that these theories are applicable to composites where the magnitudes of the material properties, especially the shear moduli, are of the same order.

The solution of the exact three-dimensional equations of elasticity has also been attempted (Jones, 1970, 1971; Srinivas *et al.*, 1970) for composite plates with rectangular planform as well as for plates with one infinitely long dimension. However, the algebra involved in three-dimensional solutions becomes too complex to be practical. Even under plane strain conditions, the exact solution becomes very complicated with the introduction of more layers in the laminate. Thus there is a necessity of developing an approach for generally anisotropic laminated plates which could result in an accuracy as substantiated by the more exact methods, yet at the same time retain the simplicity and efficiency especially when a large number of layers is being considered. The state space approach developed in this paper fills this need. The attractive features of the approach are that it does not involve any formal approximations in the three-dimensional equations of elastodynamics and the interlayer traction and displacement continuities are satisfied exactly. The boundary conditions at the lateral edges and surfaces can be prescribed either in terms of stresses or in terms of displacements or a combination of both. The approach, as formulated by Vlasov and Leontev (1966), is basically a mixed method of formulation in which both stresses and displacements are unknowns. These stresses and displacements are written in terms of their values at a chosen reference plane. The unknown initial functions at the reference plane are determined by invoking appropriate boundary conditions at another plane where some of the field variables are prescribed or zero. The resulting differential equations are independent of the derivatives with respect to one of the spatial coordinates thereby reducing the dimensionality of the problem by one. Chandrashekara (1982) and Hanagud *et al.* (1983) have successfully applied the method to study the traveling waves in layered media. The free vibration in cross-ply and angle-ply laminates has also been studied by Chandrashekara and Santhosh (1990) and Chandrashekara and Chander (1989). In these studies, good correlation has been found with the exact solutions of three-dimensional equations of elasticity. Recently, Faraji and Archer (1985, 1989) have applied the method of initial functions to the static problem of thick isotropic and transversely isotropic shells. Taylor series expansions are assumed for the field variables and approximate theories of various orders are obtained by considering a limited number of terms in the series. State space equations have been used by Jiarang and Jianqiao (1990) to solve the axisymmetric free vibration problems of transversely isotropic circular plates. Their solutions have been favorably compared with those due to the approximate theories of Reissner and Mindlin.

The state space approach has been extended in this present paper to study the free vibration response of thick angle-ply laminae and the specific results are compared with those from the exact solution presented by Jones (1971). The plate is assumed to be infinitely long in one dimension and simply supported at the lateral edges. The problem is still a three-dimensional one because of the consideration of all field variables by virtue of the nonzero elastic constants of orthotropy such as $c_{16}$, $c_{26}$, $c_{45}$ etc. However, the field variables are independent of the coordinate in the direction of infinite length. The efficiency of the method is demonstrated for various orientations of the fiber in the lamina as well as for the cases of laminae with similar and very dissimilar material properties in the directions of orthotropy. Also, a wide range of thicknesses is considered to study the efficiency of the present approach for a broad range of problems in composite dynamics. Another important feature of the present solution is that the numerical calculations are conveniently carried out on a personal computer.

## APPROACH

The basic idea of state space approach is to write a vector $\{\mathbf{q}\}$ of a set of dependent field variables in terms of a corresponding vector $\{\mathbf{q}_0\}$ at a reference or initial plane as

$$\{\mathbf{q}\}_z = [\mathbf{L}]\{\mathbf{q}_0\}. \tag{1}$$

The state vector $\{\mathbf{q}\}$ at any location $z$ and the initial vector $\{\mathbf{q}_0\}$ are related through the transfer matrix [L]. In elastodynamic problems, the state vector consists of stresses and displacements. In the above equation $z$ refers to the thickness coordinate as shown in Fig. 1 and satisfies the condition $0 \leqslant z \leqslant H/2$, where $H$ is the total thickness of the lamina. In the present case, the initial plane (i.e., $z = 0$) is chosen to be the top surface of the lamina where the stresses $\sigma_z$, $\tau_{xz}$ and $\tau_{yz}$ are known to be zero. In other words, the elements of the initial vector $\{\mathbf{q}_0\}$ are $u^0$, $v^0$ and $w^0$. The other reference plane is now chosen to be the middle surface of the lamina where some field variables are known to be zero depending on the symmetric or the antisymmetric case being considered. Thus three equations result

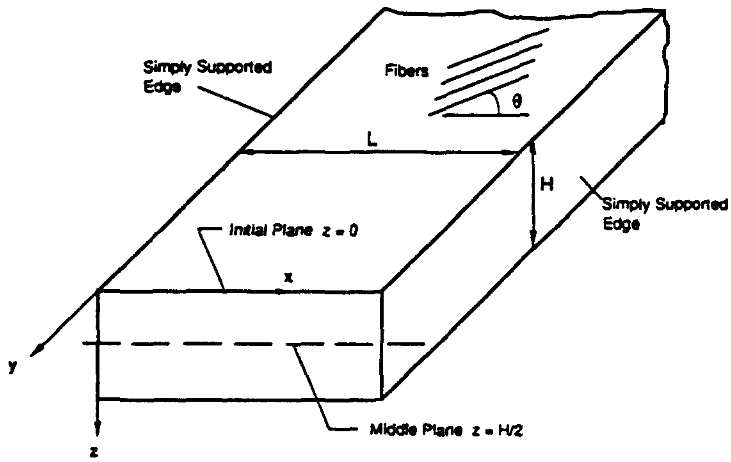


Fig. 1. Coordinate axes and geometry of the lamina.

for the three unknown initial functions which can be solved numerically to obtain the frequencies.

*Field equations*

In order to be consistent with the formulation of the exact solution (Jones, 1971), the field equations are written for the case of a lamina with simple supports on edges $x = 0$ and $x = L$ (Fig. 1) and of infinite length in the $y$-direction. The angle-ply lamina still gives rise to the displacement $v$ in the $y$-direction, but the field variables are independent of the $y$-coordinate. The small displacement kinematic relations are then written as

$$\begin{aligned}
\varepsilon_x &= \partial u/\partial x \\
\varepsilon_z &= \partial w/\partial z \\
\varepsilon_{xy} &= \partial v/\partial x \\
\varepsilon_{yz} &= \partial v/\partial z \\
\varepsilon_{xz} &= \partial w/\partial x + \partial u/\partial z
\end{aligned} \tag{2}$$

and the stress displacement relations with respect to the lamina axes are

$$\begin{aligned}
\sigma_x &= c_{11}\,\partial u/\partial x + c_{13}\,\partial w/\partial z + c_{16}\,\partial v/\partial x \\
\sigma_y &= c_{12}\,\partial u/\partial x + c_{23}\,\partial w/\partial z + c_{26}\,\partial v/\partial x \\
\sigma_z &= c_{13}\,\partial u/\partial x + c_{33}\,\partial w/\partial z + c_{36}\,\partial v/\partial x \\
\tau_{yz} &= c_{44}\,\partial v/\partial z + c_{45}(\partial w/\partial x + \partial u/\partial z) \\
\tau_{xz} &= c_{55}(\partial w/\partial x + \partial u/\partial z) + c_{45}\,\partial v/\partial z \\
\tau_{xy} &= c_{16}\,\partial u/\partial x + c_{36}\,\partial w/\partial z + c_{66}\,\partial v/\partial x.
\end{aligned} \tag{3}$$

It should be noted that the shear stresses $\tau_{yz}$ and $\tau_{xy}$ are not zero because of the angular orientation of the fiber. In eqn (3), $c_{ij}$ $(i, j = 1, 2, \ldots, 6)$ refer to the orthotropic material constants with respect to the lamina axes that are transformed from those with respect to the principal axes of orthotropy $(C_{ij})$ by an angular rotation. Finally, the equations of motion are written as

$$\begin{aligned}\partial\sigma_x/\partial x+\partial\tau_{xz}/\partial z &= \rho\,\partial^2 u/\partial t^2\\ \partial\tau_{yz}/\partial z+\partial\tau_{xy}/\partial x &= \rho\,\partial^2 v/\partial t^2\\ \partial\sigma_z/\partial z+\partial\tau_{xz}/\partial x &= \rho\,\partial^2 w/\partial t^2\end{aligned} \tag{4}$$

where $\rho$ is the density of the material.

For the purpose of analysis, the plate behavior is split up into symmetric and antisymmetric parts which yield flexural and extensional modes of vibration. Considering the middle plane of the plate as the reference plane (Fig. 1), the mid-plane conditions for the symmetric case are written as

$$z = H/2,\quad w = \tau_{xz} = \tau_{yz} = 0. \tag{5a}$$

For the antisymmetric case, they are

$$z = H/2,\quad u = v = \sigma_z = 0. \tag{5b}$$

The surface conditions at the top or the bottom of the plate are

$$z = 0,\quad \sigma_z = \tau_{xz} = \tau_{yz} = 0. \tag{5c}$$

The remaining boundary conditions for the problem have to be prescribed at the longitudinal edges represented by $x = 0$ and $x = L$.

*State space equations*

The state vector of interest in the general problem can be split into two subvectors for mathematical convenience as follows:

$$\lfloor \mathbf{q}_1 \rfloor = \lfloor u \quad v \quad \sigma_z \rfloor,\ \lfloor \mathbf{q}_2 \rfloor = \lfloor \tau_{xz} \quad \tau_{yz} \quad w \rfloor \tag{6}$$

where the symbol $\lfloor \quad \rfloor$ denotes a row vector. By a mathematical manipulation of field equations (2), (3) and (4), the following transfer matrix relations can be written for the two vectors $\{\mathbf{q}_1\}$ and $\{\mathbf{q}_2\}$:

$$\begin{aligned}\partial\{\mathbf{q}_1\}/\partial z &= [A(x,t)]\{\mathbf{q}_2\}\\ \partial\{\mathbf{q}_2\}/\partial z &= [B(x,t)]\{\mathbf{q}_1\}\end{aligned} \tag{7}$$

where $[A(x,t)]$ and $[B(x,t)]$ are given by

$$[A] = \begin{bmatrix} \dfrac{c_{44}}{d_1} & \dfrac{-c_{45}}{d_1} & -\alpha \\ \dfrac{-c_{45}}{d_1} & \dfrac{c_{55}}{d_1} & 0 \\ -\alpha & 0 & \rho\xi^2 \end{bmatrix}$$

$$[B] = \begin{bmatrix} \rho\xi^2-\{c_{11}-(c_{13}^2/c_{33})\}\alpha^2 & -\{c_{16}-(c_{13}c_{36}/c_{33})\}\alpha^2 & -(c_{13}/c_{33})\alpha \\ -\{c_{16}-(c_{13}c_{36}/c_{33})\}\alpha^2 & \rho\xi^2-\{c_{66}-(c_{36}^2/c_{33})\}\alpha^2 & -(c_{36}/c_{33})\alpha \\ -(c_{13}/c_{33})\alpha & -(c_{36}/c_{33})\alpha & 1/c_{33} \end{bmatrix}.$$

In these equations and in what follows, the following notations are used:

$$d_1 = (c_{44}c_{45}-c_{45}^2);\quad \alpha = \partial/\partial x;\quad \alpha^2 = \partial^2/\partial x^2,\quad \alpha^4 = \partial^4/\partial x^4 \text{ etc.}$$

and

$$\xi^2 = \partial^2/\partial t^2; \quad \xi^4 = \partial^4/\partial t^4 \text{ etc.}$$

The solution of eqn (7) is sought in terms of a Maclaurin series in the $z$-direction for the state variables. These are written, for example, for displacement $u$, as

$$u(x, z, t) = u^0 + \frac{z^n}{n!} \sum_{i=0}^{\infty} \left( \frac{\partial^n u}{\partial z^n} \right)^0. \tag{8}$$

The superscript "0" in the above equation refers to the surface $z = 0$ for the layer $i$, and $u^0 = u^0(x, t)$. Thus the initial vector to be determined is a function of only one spatial coordinate and time. By using eqn (7) successively, it is possible to write the higher order derivatives of vectors $\{\mathbf{q}_1\}$ and $\{\mathbf{q}_2\}$ with respect to $z$. The series expansions for the field variables as indicated in eqn (8) can be written as

$$\left\{ \begin{matrix} u \\ v \\ \sigma_z \\ \tau_{xz} \\ \tau_{yz} \\ w \end{matrix} \right\} \text{ at any } z = \begin{bmatrix} L_{uu} & L_{uv} & L_{uz} & L_{ux} & L_{uy} & L_{uw} \\ L_{vu} & L_{vv} & L_{vz} & L_{vx} & L_{vy} & L_{vw} \\ L_{zu} & L_{zv} & L_{zz} & L_{zx} & L_{zy} & L_{zw} \\ L_{xu} & L_{xv} & L_{xz} & L_{xx} & L_{xy} & L_{xw} \\ L_{yu} & L_{yv} & L_{yz} & L_{yx} & L_{yy} & L_{yw} \\ L_{wu} & L_{wv} & L_{wz} & L_{wx} & L_{wy} & L_{ww} \end{bmatrix} \left\{ \begin{matrix} u \\ v \\ \sigma_z \\ \tau_{xz} \\ \tau_{yz} \\ w \end{matrix} \right\} \text{ at } z = 0. \tag{9}$$

Equation (9) can be seen to have the same form as eqn (1). The $L$ values appearing here are the derivative operators (in terms of $x$ and $t$ only) of various powers depending on the number of terms considered in the Maclaurin series expansion. The subscripts on these operators refer to the matrix elements linking the field variables at any $z$ to the field variables at the initial surface $z = 0$. It should be noted that these equations refer to a single layer.

With $u^0$, $v^0$ and $w^0$ as the initial functions at the surface $z = 0$ and invoking the mid-surface conditions given in eqn (5a), eqn (9) can be reduced to the final differential equations for the symmetric case as follows:

$$\begin{aligned} L_{xu}u^0 + L_{xv}v^0 + L_{xw}w^0 &= 0 \\ L_{yu}u^0 + L_{yv}v^0 + L_{yw}w^0 &= 0 \\ L_{wu}u^0 + L_{wv}v^0 + L_{ww}w^0 &= 0. \end{aligned} \tag{10a}$$

In a similar way, the antisymmetric mid-surface conditions given in eqn (5b) can be applied to the general equations (9) to obtain the final differential equations for this case as

$$\begin{aligned} L_{uu}u^0 + L_{uv}v^0 + L_{uw}w^0 &= 0 \\ L_{vu}u^0 + L_{vv}v^0 + L_{vw}w^0 &= 0 \\ L_{zu}u^0 + L_{zv}v^0 + L_{zw}w^0 &= 0. \end{aligned} \tag{10b}$$

The operator functions in eqn (10) now contain only the derivatives with respect to $x$ and $t$ and are written with $z = H/2$. By considering a finite number of terms in the series represented by eqn (8) for the field variables, various order theories of the present state space formulation based on the exact field equations can be obtained. As a definition, a $k$th order theory is one in which terms up to $z^k$ are considered. The solutions for the unknowns in eqn (10) are sought by means of a separation of variables technique. The resulting characteristic equations can be solved to give the frequencies of vibration.

## NUMERICAL SOLUTION AND DISCUSSION

The exact solution for the free vibration of a thick off-axis lamina has been given by Jones (1971) for a laminate with finite $x$-dimension. The boundary conditions at the simply supported edges are defined as

$$\sigma_x(x,z,t) = 0; \quad \tau_{xy}(x,z,t) = 0 \quad \text{and} \quad w(x,z,t) = 0. \tag{11}$$

It can be easily verified that these boundary conditions give rise to a consistent set of boundary condition equations for the order of theory chosen for analysis. The equations for $\sigma_x$ and $\tau_{xy}$ are derived from state space equations (9) and the stress–displacement relations (3). To satisfy the conditions (11), the solutions for $u^0$, $v^0$ and $w^0$ can be assumed in the following form:

$$u^0(x,t) = \sum_{n=1}^{\infty} U_n \cos(n\pi x/L)\, e^{i\omega_n t}$$

$$v^0(x,t) = \sum_{n=1}^{\infty} V_n \cos(n\pi x/L)\, e^{i\omega_n t}$$

$$w^0(x,t) = \sum_{n=1}^{\infty} W_n \sin(n\pi x/L)\, e^{i\omega_n t} \tag{12}$$

where $U_n$, $V_n$ and $W_n$ are constants and $\omega_n$ is the frequency of vibration. Substitution of eqn (12) into eqn (10) and setting the determinant to zero gives rise to a characteristic or frequency equation which when solved gives the frequencies of natural vibration.

In the subsequent presentation of results, to be consistent with the notations of the exact solution by Jones (1971), the thickness ratio is defined to be the ratio of the total thickness of the lamina to the length or the span between the supports ($nH/L$). The non-dimensional frequency $\omega^*$ is defined as

$$\omega^* = \sqrt{\rho\omega_n^2/p^2 C_{11}}$$

where $\rho$ and $C_{11}$ refer to the density and stiffness constant in the principal direction respectively and the parameter $p = n\pi/L$. In the following presentations, the non-dimensional frequencies are plotted against the thickness parameter.

The material properties chosen for the lamina are given in Table 1. As can be seen from this table, the two materials chosen for analysis represent cases of similar and very dissimilar properties. The ratios of longitudinal to transverse moduli for the two materials are 4 and 40, indicating that the material under case 1 is less stiff than the other. The numerical solutions presented here refer to these two cases. Figures 2a and 2b show the results of convergence studies for a 15° lamina with material 2 for the flexural and extensional modes respectively. The results from the 4th, 8th, 12th and 16th order theories are shown for the lowest three fundamental frequencies. As can be seen from these figures, an excellent convergence is achieved for the entire range of thickness parameters with a 16th order theory and hence, in all the following numerical calculations, a 16th order theory is considered.

Figures 3–7 show the variations of the nondimensional frequencies $\omega^*$ against the nondimensional thickness parameter (or the wave number parameter) $nH/L$ for lamina

Table 1.

| | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{22}$ | $C_{23}$ | $C_{33}$ | $C_{44}$ | $C_{55}$ | $C_{66}$ |
|---|---|---|---|---|---|---|---|---|---|
| Case 1 | 4.0 | 0.25 | 0.25 | 1.0 | 0.3 | 1.0 | 0.5 | 0.6 | 0.6 |
| Case 2 | 40.0 | 0.25 | 0.25 | 1.0 | 0.3 | 1.0 | 0.5 | 0.6 | 0.6 |

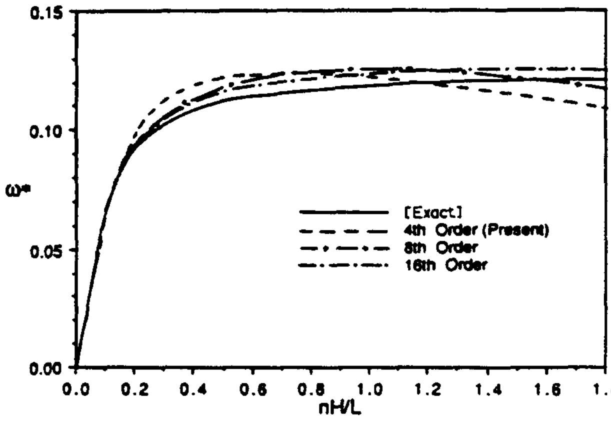


Fig. 2a. Convergence of frequency results for the first mode of vibration with various orders (15° lamina with material 2).

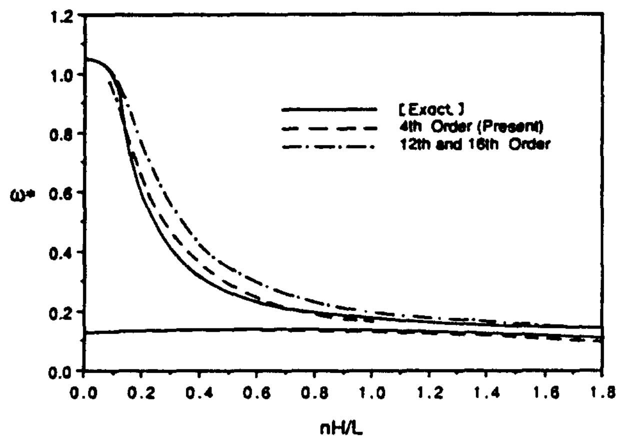


Fig. 2b. Convergence of frequency results for the second and third modes of vibration with various orders (15° lamina with material 2).

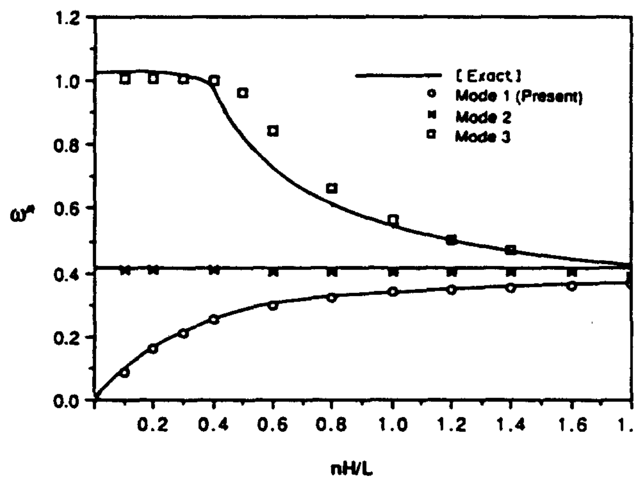


Fig. 3a. Variation of nondimensional frequencies with wave number parameters (15° lamina, material 1).

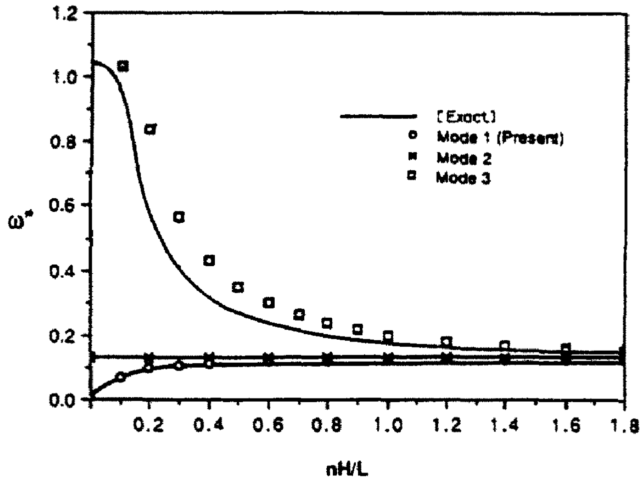


Fig. 3b. Variation of nondimensional frequencies with wave number parameters (15° lamina, material 2).

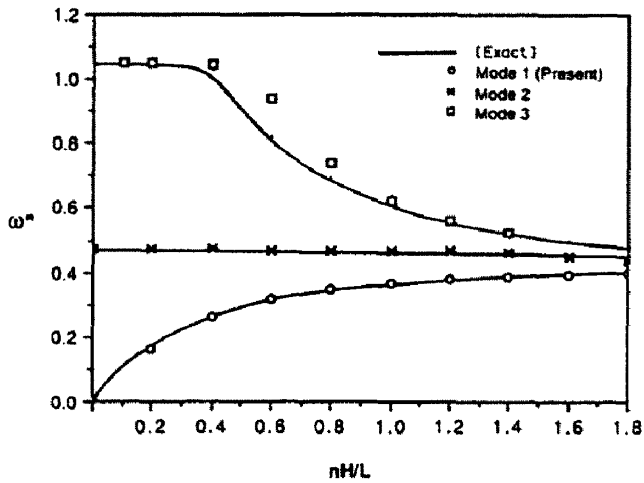


Fig. 4a. Variation of nondimensional frequencies with wave number parameters (30° lamina, material 1).

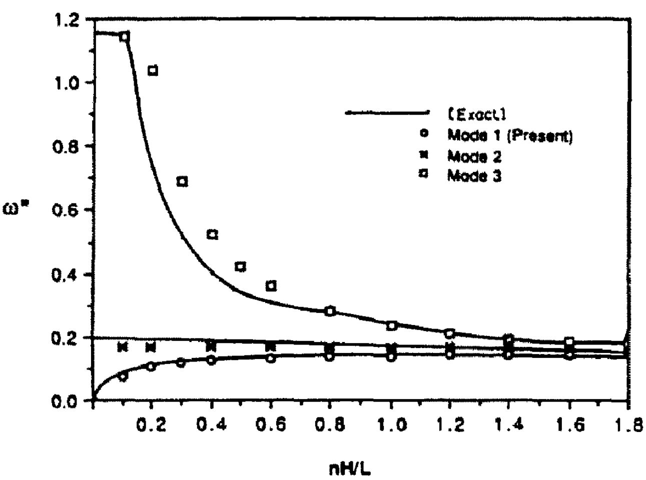


Fig. 4b. Variation of nondimensional frequencies with wave number parameters (30° lamina, material 2).

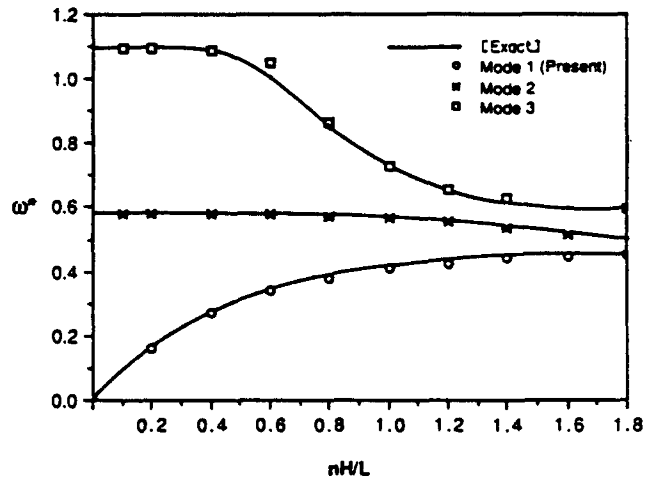


Fig. 5a. Variation of nondimensional frequencies with wave number parameters (45° lamina, material 1).

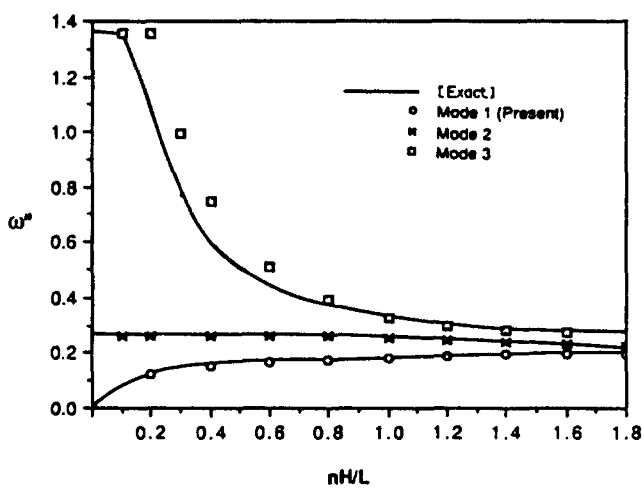


Fig. 5b. Variation of nondimensional frequencies with wave number parameters (45° lamina, material 2).

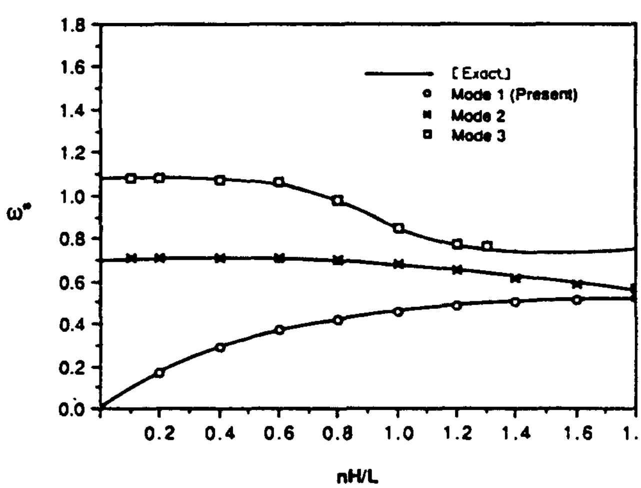


Fig. 6a. Variation of nondimensional frequencies with wave number parameters (60° lamina, material 1).

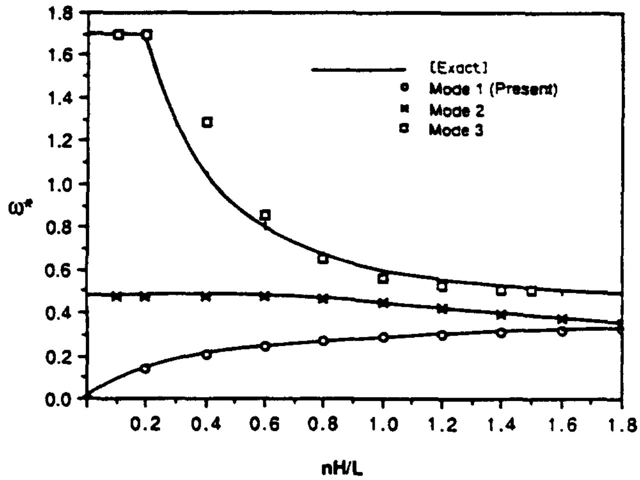


Fig. 6b. Variation of nondimensional frequencies with wave number parameters (60 lamina, material 2).

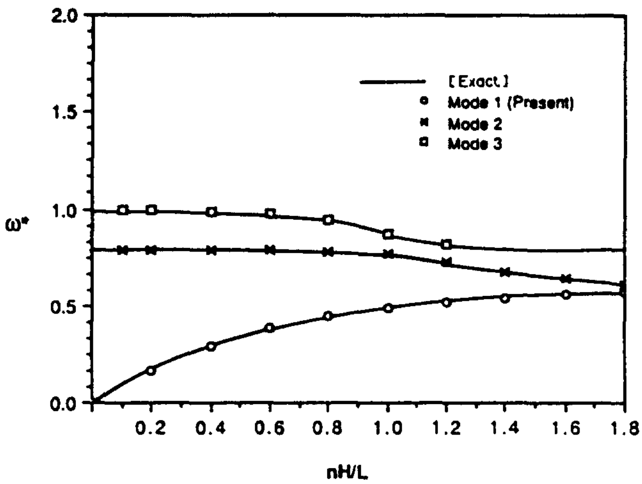


Fig. 7a. Variation of nondimensional frequencies with wave number parameters (75 lamina, material 1).

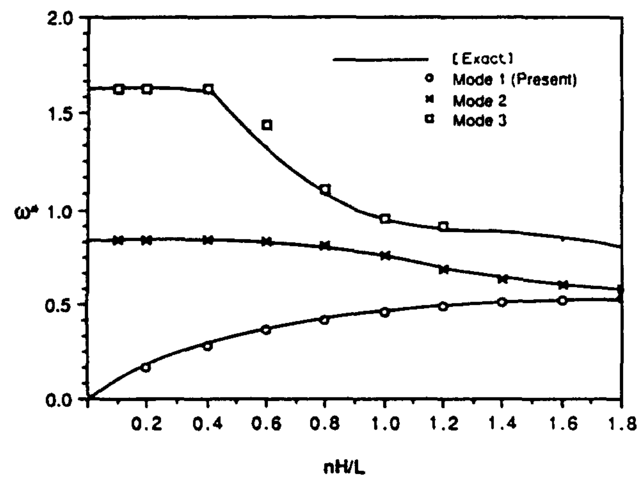


Fig. 7b. Variation of nondimensional frequencies with wave number parameters (75 lamina, material 2).

inclination angles of 15°, 30°, 45°, 60° and 75°. Figures 3a–7a represent material system 1 with similar material properties while Figs 3b–7b represent the material system 2 with very dissimilar material properties of the lamina. The thickness parameter includes both "thin" plate and very thick plates. The maximum value of the thickness ratio chosen in all the results is $nH/L = 1.8$. The results from the classical plate theory are not shown here since it has already been well established that they are valid only for thin plates characterized by the ratio $nH/L$ being less than about 0.1. This is especially true of the flexural frequencies in all the cases. The correlation of the classical plate theory solutions with the exact solutions is more accurate for a broader range of thickness parameters for the first extensional frequencies. These and other relevant discussions regarding the applicability of the classical plate theory are well documented by Jones (1971). The present discussion deals with the comparisons of the results from the state space approach with those from the exact solution.

The comparisons of nondimensional frequencies from the present approach with those from the exact solution are shown in Figs 3–7. An excellent correlation can be observed for the flexural and the first extensional frequencies for all the cases under consideration even for very thick plates characterized by the ratio $nH/L = 1.8$. In other words, the present approach yields results that are acceptable over a broad range of anisotropy of the lamina as well as broad range of wavelengths. A fair comparison for the third frequency for the various inclination angles can also be observed from these figures. The correlation of the results from the state space approach with those from the exact solution tends to become more accurate for the third frequency as the anisotropic angle increases from 15° to 75°. In general, it can be observed from the results for the first three modes of vibration that the state space approach gives acceptable results for a wide range of anisotropy angles and for very high thickness values of the lamina.

From the standpoint of numerical calculations, it is worth mentioning that all the calculations were run on a personal computer. Simple subroutines for multiplication of derivative matrices as well as for equation solvers have been adopted.

*Extension of the approach to a laminate*

It is possible to extend the present formulation to the case of a layered plate with different material properties assigned to the layers. Equations expressing the basic idea of the state space approach, i.e., eqns (1) or (9), can now be written for the case when there are $n$ layers. These relationships will be of the form

$$\{\mathbf{q}^n\}_{Z_n} = \{\mathbf{L}^n]_{H_n}[\mathbf{L}^{n-1}]_{H_{n-1}} \cdots [\mathbf{L}^2]_{H_2}[\mathbf{L}^1]_{H_1}\{\mathbf{q}^0\} \tag{13}$$

where $\{\mathbf{q}^n\}$ and $[\mathbf{L}^n]$ are the state vector and the field matrix, respectively, for the layer $n$ (for any $z$ in that layer) and $[\mathbf{L}^{n-1}]_{H_{n-1}}$ is the transfer matrix for the layer $n-1$ with the values for $z$ for that layer being substituted by its thickness $H_{n-1}$ and so on up to layer 1. Equations (13) are also valid for the surface $z_n = H_n$. With the coordinate axes similar to the ones shown in Fig. 1, the initial plane and the final reference plane for each layer can be chosen to be the top and bottom surfaces, respectively, of that layer. The free surface conditions on the initial plane $z_0 = 0$ for the top-most layer and the final reference plane $z_n = H_n$ for the bottom-most layer result in a set of differential equations given by eqn (13) which can be solved for the remaining unknown field variables at the initial plane. It is important to note that the continuity conditions at their interfaces between layers are automatically and exactly satisfied because the same principle of transfer matrix is used to eliminate all the intermediate state vectors of field variables representing those intermediate layers. The state vector for any layer is then determined by the product of the initial vector times a chain of transfer matrices of the individual layer times the field matrix of the layer of interest. In addition, the state space approach as described in this paper can be extended to study the vibration characteristics of plates with finite dimensions (without plane strain assumptions) as well as wave propagation problems in composite layered media. These studies are currently under investigation and will be the subject matters of further research papers in the future.

## CONCLUSIONS

The state space approach, besides being simple mathematically, has been shown to have an excellent capability of predicting the natural frequencies of vibration of angle-ply laminae even with large thicknesses, as well as for laminae with very dissimilar material properties. The computations can be handled on a personal computer and theories of various orders can be numerically solved with ease depending on the required accuracy for the particular problem. The approach gives results which are in very good agreement with those from the exact solution of three-dimensional elasticity equations for a broad range of anisotropy angles and a wide range of thickness ratios. The consideration of a large number of layers in the case of an anisotropic laminate can be easily effected by a series of transfer matrices and the corresponding multiplications through simple computer routines.

*Acknowledgements*—The authors would like to thank Dr. Som Soni of AdTech Systems Research, Inc. and Mr. Clark Graham of Advanced Composite Technology, Inc. for supporting the preparation of this paper.

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